UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1201**

ASSESSMENT PATTERN : **MATH1201A**

MODULE NAME : **Algebra 1**

DATE : **29-Apr-08**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2007/08-MATH1201A-001-EXAM-190

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (i) Replace the following by an equivalent formula which does *not* involve $\neg$ or $\Longrightarrow$;

$$[(\forall y)Q(y) \Longrightarrow (\forall x)\neg P(x)] \implies [(\exists y)\neg R(y) \Longrightarrow \neg(\exists x)\neg P(x)]\,.$$

(ii) Let $f : A \to B$ be a mapping between sets A, B. Explain what is meant by saying that f is *injective* ; f is *surjective* ; f is *invertible.*

Prove that if f is both injective and surjective then f is invertible.

In each case below decide whether the given mapping f is injective and also whether f is surjective; moreover, if f is bijective, give the explicit form of f^{-1}:

(a) $f : \mathbb{Z} \to \mathbb{Z}$; $f(x) = 4x + 1$;

(b) $f : \mathbb{Q} \to \mathbb{Q}$; $f(x) = 4x + 1$;

(c) $f : \mathbb{C} \to \mathbb{C}$; $f(x) = 4x^2 + 1$.

2. Let $\epsilon(r, s)$ be the basic $m \times m$ matrix given by $\epsilon(r, s)_{ij} = \delta_{ri}\delta_{sj}$ where 'δ' denotes the Kronecker delta. Describe in detail the elementary $m \times m$ matrices

(i) $E(r, s; \lambda)$ $(r \neq s)$; (ii) $\Delta(r, \lambda)$ $(\lambda \neq 0)$; (iii) $P(r, s)$ $(r \neq s)$

in terms of the basic matrices $\epsilon(r, s)$.

Furthermore, show by a calculation that $E(r, s; \lambda)$ is invertible for $r \neq s$ and write down its inverse.

For the matrix A below, find A^{-1} and express A^{-1} as a product of elementary matrices; hence also express A as a product of elementary matrices.

$$A = \begin{pmatrix} -1 & 2 & -4 \\ -1 & 1 & -2 \\ 1 & -1 & 3 \end{pmatrix}.$$

3. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V; explain what is meant by saying that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

In each case below, decide with justification whether the given vectors are linearly independent. If they are not, give an explicit dependence relation between them.

(a) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ -1 \end{pmatrix};$

(b) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$

Explain what is meant by a *spanning set* for a vector space V. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a spanning set for V, and suppose that $\mathbf{u} \in V$ can be expressed as a linear combination of the form

$$\mathbf{u} = \sum_{r=1}^{n} \lambda_r \mathbf{v}_r$$

with $\lambda_1 \neq 0$. Show that $\{\mathbf{u}, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is also a spanning set for V.

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a linearly independent set show that there exists a spanning set $\{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ in which $\mathbf{v}'_i = \mathbf{u}_i$ for $1 \leqslant i \leqslant k$.

4. Let V, W be vector spaces over a field $\mathbb{F}$ and let $T : V \to W$ be a mapping; explain what is meant by saying that T is *linear*.

When T is linear, explain what is meant by

(a) the kernel, $\mathrm{Ker}(T)$ and

(b) the image, $\mathrm{Im}(T)$.

State and prove a relationship which holds between $\dim \mathrm{Ker}(T)$ and $\dim \mathrm{Im}(T)$.

Let $T_A : \mathbb{Q}^6 \to \mathbb{Q}^4$ be the linear mapping $T_A(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 2 & -1 \end{pmatrix}.$$

Find (i) $\dim \mathrm{Ker}(T_A)$; (ii) a basis for $\mathrm{Ker}(T_A)$; (iii) a basis for $\mathrm{Im}(T_A)$.

5. (i) Let σ be a permutation of the set $\{1,\ldots,n\}$; explain what is meant by saying that σ is (a) a *transposition* ; (b) an *adjacent transposition*.

Show that any transposition can be written as a product of an odd number of adjacent transpositions.

$$\text{Decompose}\quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 8 & 11 & 10 & 12 & 14 & 2 & 13 & 5 & 4 & 3 & 7 & 9 & 6 & 1 \end{pmatrix}$$

into a product of disjoint cycles and hence compute $\mathrm{sign}(\sigma)$ and $\mathrm{ord}(\sigma)$.

(ii) Let $\mathcal{P}_{11}(\mathbb{R})$ be the vector space of polynomials of degree $\leqslant 11$ over the field $\mathbb{R}$ and let $D : \mathcal{P}_{11}(\mathbb{R}) \to \mathcal{P}_{11}(\mathbb{R})$ be the linear map given by differentiation. Write down the least positive integer n for which $D^{2n} = 0$ on $\mathcal{P}_{11}(\mathbb{R})$.

By factorisation of the formal expressions $D^{2n} - \mathrm{I}$, $D^n - \mathrm{I}$ or otherwise, show that the mapping

$$D^9 - D^6 + D^3 - \mathrm{I} \; : \; \mathcal{P}_{11}(\mathbb{R}) \;\to\; \mathcal{P}_{11}(\mathbb{R})$$

is invertible, and write down an expression for its inverse in terms of D. Hence find the unique solution $\alpha \in \mathcal{P}_{11}(\mathbb{R})$ to the differential equation

$$\frac{d^9\alpha}{dx^9} - \frac{d^6\alpha}{dx^6} + \frac{d^3\alpha}{dx^3} - \alpha \;=\; -x^7 - x^4.$$

6. Let $T : U \to V$ be a linear map between vector spaces U , V, and let $\mathcal{E} = (e_i)_{1\leqslant i\leqslant n}$ be a basis for U and $\Phi = (\varphi_j)_{1\leqslant j\leqslant m}$ be a basis for V. Explain what is meant by the matrix $m(T)^{\Phi}_{\mathcal{E}}$ of T taken with respect to $\mathcal{E}$ (on the left) and Φ (on the right) and prove that if $S : V \to W$ is also a linear map and $\Psi = (\psi_k)_{1\leqslant k\leqslant p}$ is a basis for W then

$$m(S\circ T)^{\Psi}_{\mathcal{E}} = m(S)^{\Psi}_{\Phi}\, m(T)^{\Phi}_{\mathcal{E}}.$$

$$\text{Let } \mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \;;\quad \Phi = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

be bases for $\mathbf{F}^3$ and let $S : \mathbf{F}^3 \to \mathbf{F}^3$ be the mapping

$$S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 & -2x_2 & \\ & x_2 & +2x_3 \\ & & x_3 \end{pmatrix}.$$

Write down (i) $m(T)^{\mathcal{E}}_{\mathcal{E}}$ and (ii) $m(\mathrm{Id})^{\mathcal{E}}_{\Phi}$, and hence find $m(T)^{\Phi}_{\Phi}$.

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